Technical Comments

Comment on "Theory of Thin Airfoils in Magnetoaerodynamics"

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In a paper[1] examining the motion of a compressible fluid with arbitrary finite electrical conductivity in the presence of a thin airfoil, the author derived the equations

$$(1 - M^2)\partial u/\partial x + \partial v/\partial y = M^2\xi/A^2$$

$$\partial v/\partial x - \partial u/\partial y = \omega$$

where ξ and ω represent, respectively, the "curl" of the magnetic and velocity fields. The solution is taken in the form of the general solution of the homogeneous system added to a particular solution of the nonhomogeneous system. This procedure leads to three equations (4.4, 5.4, and 6.12) for two functions.

These three equations are inconsistent. The error occurs in that ξ and ω are not independent functions but are related to u and v through the magnetoaerodynamic equations (2.4–2.8).

Reference

[1] Dragos, L., "Theory of thin airfoils in magnetoaerodynamics," AIAA J. 2, 1223–1229 (1964).

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Reply by Author to P. Greenberg

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Following the objection of P. Greenberg that in our paper[1] we have obtained three equations (4.4, 5.4, 6.12) for two functions and since these equations are inconsistent, we present a new solution of the problem—based essentially on the idea of the first paper—which shows that the error indicated does not exist.

It is shown in particular that the two functions φ and ψ must vanish. Accordingly the three equations (4.4, 5.4, 6.12) can exist. Besides, if in Ref. 1 the complete reasoning had been made, it would have resulted that the two functions must vanish.

Under the conditions as in Ref. 1 and using almost the same notations we have to integrate the following system,

$$M^2(\partial p/\partial x) + \mathrm{div}\mathbf{v} = 0 \quad (1)$$

$$\partial \mathbf{v}/\partial x = -\mathrm{grad}p + (1/A^2)\mathbf{J} \times \mathbf{e}_2 \quad (2)$$

$$\mathrm{rot}\mathbf{h} = \mathbf{J} = J\mathbf{e}_3 = R_M(\mathbf{E}_1 + \mathbf{e}_1 \times \mathbf{h} + \mathbf{v} \times \mathbf{e}_2) \quad (3)$$

$$\mathrm{rot}\mathbf{E}_1 = 0 \quad (4)$$

$$\mathrm{div}\mathbf{h} = 0 \quad (5)$$

Since the motion is a plane one from (3) and (4), there results $\mathbf{E}_1 = 0$.

From (3) we deduce the equation of magnetic induction

$$R_M^{-1}\,\Delta\mathbf{h} = \frac{\partial \mathbf{h}}{\partial x} - \frac{\partial \mathbf{v}}{\partial y} + \mathbf{e}_2\,\mathrm{div}\mathbf{v} \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (6)$$

By applying the operator rot in Eq. (2) we get

$$\partial\omega/\partial x = (1/A^2)(\partial J/\partial y) \quad (7)$$

By derivation with respect to the variable x of Eq. (2) and using Eq. (1), we eliminate the pressure. By applying then the operator div to the equation found, we have

$$H\theta = \left[(1 - M^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\theta = \frac{M^2}{A^2}\frac{\partial^2 J}{\partial x^2} \quad (8)$$

Finally, applying the operator rot in Eq. (6) we obtain

$$PJ = \left(R_M^{-1}\Delta - \frac{\partial}{\partial x}\right)J = -\frac{\partial\omega}{\partial y} + \frac{\partial\theta}{\partial x} \quad (9)$$

We have used the following notations:

$$\mathrm{div}\mathbf{v} = \theta \qquad \mathrm{rot}\mathbf{v} = \omega\mathbf{e}_3 \quad (10)$$

From Eqs. (7–9) we have

$$T\theta = T\omega = TJ = 0 \quad (11)$$

where T is the operator

$$T = N^{-1}\Delta H\frac{\partial}{\partial x} + (A^2M^2 - A^2 - M^2)\frac{\partial^4}{\partial x^4} - (A^2 + M^2 - 1)\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4} \quad (12)$$

with the following notation

$$R_M = A^2N \quad (13)$$

N being the parameter of the magnetohydrodynamic interaction.

The fact that the three unknowns θ, ω, and J satisfy the same equation (11) is of a particular importance for determining the solution of the problem.

We shall look for the general solution of Eq. (11) under the form of a continuous superposition of plane waves of the form

$$\exp[-i\lambda(x + sy)] \quad (14)$$

By replacing in (11) and (12) we get

$$as^4 - bs^2 + c = 0 \quad (15)$$

where

$$\left.\begin{aligned} a &= 1 - (i\lambda/N) \\ b &= A^2 + M^2 - 1 - (i\lambda/N)(M^2 - 2) = b_1 - (i\lambda/N)b_2 \\ c &= A^2M^2 - A^2 - M^2 - (i\lambda/N)(1 - M^2) = c_1 - (i\lambda/N)c_2 \end{aligned}\right\} \quad (16)$$

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The roots of Eq. (15) are complex and have the imaginary part different from 0. If this assertion were true, it would result that the equation

$$a(R + iI)^4 - b(R + iI)^2 + c = 0 \tag{17}$$

would have at least one real root. It may be deduced that the equations

$$R^4 - b_1R^2 + c_1 = 0 \qquad R^4 - b_2R^2 + c_2 = 0 \tag{18}$$

would have a common real root. However the second equation of (18), which corresponds in fact to the 5th-order terms in the expression of the operator T, may be easily solved. Its only real roots are $R = \pm(M^2 - 1)^{1/2}$, $M > 1$. It may be easily verified that these roots cannot be also the roots of the first equation (18).

Since Eq. (15) is biquadratic, it has two distinct roots with positive imaginary part (denoted below by s_1 and s_2) and two roots ($-s_1$ and $-s_2$) with negative imaginary part.

Taking into account that the perturbations must vanish for $y \to \pm\infty$, there results that in the upper half-plane ($y > 0$) we have waves of the form

$$\exp[-i\lambda(x - s_1y)] \qquad \exp[-i\lambda(x - s_2y)] \tag{19}$$

and in the lower half-plane ($y < 0$) we have waves of the form

$$\exp[-i\lambda(x + s_1y)] \qquad \exp[-i\lambda(x + s_2y)] \tag{20}$$

The general solution of Eq. (11) may be written as

$$\theta_\pm(x, y) = \int_{-\infty}^{+\infty} \sum_{k=1,2} \bar{\theta}_{k\pm}(\lambda)e^{\pm i\lambda s_k y}\, e^{-i\lambda x}\, d\lambda \tag{21}$$

$$\omega_\pm(x, y) = \int_{-\infty}^{+\infty} \sum_{k=1,2} \bar{\omega}_{k\pm}(\lambda)e^{\pm i\lambda s_k y}\, e^{-i\lambda x}\, d\lambda \tag{22}$$

$$J_\pm(x, y) = \int_{-\infty}^{+\infty} \sum_{k=1,2} \bar{J}_{k\pm}(\lambda)e^{\pm i\lambda s_k y}\, e^{-i\lambda x}\, d\lambda \tag{23}$$

The first $\pm$ sign indicates the solution valid in the upper half-plane, and the second one the solution valid in the lower half-plane. The twelve functions of λ that intervene in the representation of the general solution (21–23) are to be determined from the motion equations and the boundary conditions. We must also impose the condition that the solution (21–23) should verify the motion equations since Eqs. (11) were obtained from the motion equations by derivations.

By imposing condition (7) we get

$$\frac{A^2\bar{\omega}_{1\pm} \pm s_1\bar{J}_{1\pm}}{A^2{}_2\bar{v}_\pm \pm s_2\bar{J}_{2\pm}} = -e^{\pm i\lambda y(s_2 - s_1)} \tag{24}$$

Since Eq. (24) must be verified foı an arbitrary y and since the roots s_1 and s_2 are distinct, we have

$$A^2\bar{\omega}_{k\pm} = \mp s_k\bar{J}_{k\pm} \qquad k = 1, 2 \tag{25}$$

In a similar manner from Eq. (8) we deduce

$$A^2(1 - M^2 + s_k^2)\bar{\theta}_{k\pm} = M^2\bar{J}_{k\pm} \qquad k = 1, 2 \tag{26}$$

and from (9) we get

$$\bar{\theta}_{k\pm} \pm s_k w_{k\pm} = -[1 + i\lambda R_M^{-1}(1 + s_k^2)]\bar{J}_{k\pm} \qquad k = 1, 2 \tag{27}$$

Equations (25) and (26) will determine the functions $\bar{\omega}_{k\pm}(\lambda)$ and $\theta_{k\pm}(\lambda)$ with the aid of the functions $\bar{J}_{k\pm}(\lambda)$. It may be easily verified that Eq. (27) is a consequence of Eqs. (25, 26, and 15).

The velocity field may be determined from (10). We have

$$u = u_P + (\partial\varphi/\partial x) \qquad v = v_P + (\partial\varphi/\partial y) \tag{28}$$

$$\Delta\varphi = 0 \tag{29}$$

$$u_P(x, y) = \int_{-\infty}^{+\infty} \sum_{k=1,2} \bar{u}_{k\pm}e^{\pm i\lambda s_k y}\, e^{-i\lambda x}\, d\lambda$$
$$v_P(x, y) = \int_{-\infty}^{+\infty} \sum_{k=1,2} \bar{v}_{k\pm}e^{\pm i\lambda s_k y}\, e^{-i\lambda x}\, d\lambda \tag{30}$$

where

$$-i\lambda(1 + s_k^2)\bar{u}_{k\pm} = \bar{\theta}_{k\pm} \pm s_k\bar{\omega}_{k\pm}$$
$$-i\lambda(1 + s_k^2)\bar{v}_{k\pm} = \bar{\omega}_{k\pm} \pm s_k\bar{\theta}_{k\pm} \tag{31}$$

or, taking account of (25), (26),

$$i\lambda A^2(1 - M^2 + s_k^2)\bar{u}_{k\pm} = (s_k^2 - M^2)\bar{J}_{k\pm}$$
$$i\lambda A^2(1 - M^2 + s_k^2)\bar{v}_{k\pm} = \pm s_k\bar{J}_{k\pm} \tag{31'}$$

Here and in the following the subscript, k assumes values 1 and 2.

In a similar manner the magnetic field is determined from Eqs. (3) and (5). We have

$$h_x = h_{xP} + (\partial\psi/\partial x) \qquad h_y = h_{yP} + (\partial\psi/\partial y) \tag{32}$$

$$\Delta\psi = 0 \tag{33}$$

$$h_{xP}(x, y) = \int_{-\infty}^{+\infty} \sum_{k=1,2} \bar{h}_{xk\pm}e^{\pm i\lambda s_k y}\, e^{-i\lambda x}\, d\lambda$$
$$h_{yP}(x, y) = \int_{-\infty}^{+\infty} \sum_{k=1,2} \bar{h}_{yk\pm}e^{\pm i\lambda s_k y}\, e^{-i\lambda x}\, d\lambda \tag{34}$$

with the notations

$$-i\lambda(1 + s_k^2)\bar{h}_{xk\pm} = \pm s_k\bar{J}_{k\pm}$$
$$-i\lambda(1 + s_k^2)\bar{h}_{yk\pm} = \bar{J}_{k\pm} \tag{35}$$

Equations (26, 31, 31′, and 35) as well as all those given henceforth impose the conditions $s_k^2 + 1 \neq 0$ and $s_k^2 + 1 - M^2 \neq 0$. It may be easily verified that these conditions are satisfied if $M \neq 0$ and $A \neq 0$, which is assumed in the present paper. There results that the previous solution cannot be adopted to the case of the incompressible fluid ($M = 0$).

A second observation to be made refers to the damping condition at infinity. We must have

$$\lim_{|X|\to\infty} (u, v, h_x, h_y) = 0 \tag{36}$$

However, taking account of the fact that u_P, v_P, h_{xP}, h_{yP} are represented with the aid of integrals of Fourier type, on the basis of Lebesque's theorem [2] we have

$$\lim_{|X|\to\infty} (u_P, v_P, h_{xP}, h_{yP}) = 0 \tag{37}$$

Hence from (28) and (32) there results

$$\lim_{|X|\to\infty} (\partial\varphi/\partial x, \partial\varphi/\partial y, \partial\psi/\partial x, \partial\psi/\partial y) = 0 \tag{38}$$

We impose the condition that the solution found should verify the equation of the magnetic induction (6). We obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x}\right) = 0 \tag{39}$$

By using condition (38) there results

$$\frac{\partial\varphi}{\partial y} = \frac{\partial\psi}{\partial x} \tag{40}$$

By imposing Ohm's law (3),

$$J = R_M(u + h_y) \tag{3'}$$

and using also the representation of the general solution, we get

$$\partial\varphi/\partial x = -(\partial\psi/\partial y) \tag{41}$$

Conditions (40) and (41) show that the functions ψ and φ are harmonically conjugated.

On projecting Eq. (2) on the axes of the reference system, we obtain

$$(\partial u/\partial x) + (\partial p/\partial x) = -(1/A^2)J \qquad (\partial v/\partial x) + (\partial x/\partial y) = 0 \tag{2'}$$

From (1) and the first equation (2′), we deduce

$$(1 - M^2)(\partial u/\partial x) + (\partial v/\partial y) = (M^2/A^2)J \tag{1'}$$

Taking account of (28, 23, 31′, and 35), we obtain

$$(1 - M^2)(\partial^2\varphi/\partial x^2) + (\partial^2\varphi/\partial y^2) = 0$$

Since the function φ is also a harmonic one (29), there results

$$\partial^2\varphi/\partial x^2 = \partial^2\varphi/\partial y^2 = 0$$

By using the damping condition (38), we deduce

$$\partial\varphi/\partial x = \partial\varphi/\partial y = 0 \tag{42}$$

and from (40) and (41) we get

$$\partial\psi/\partial x = \partial\psi/\partial y = 0 \tag{43}$$

The results (42) and (43) demonstrate the assertion made in the introductory part in conjunction with the fact that equations (4.4., 5.4, 6.12) given in Ref. 1 can subsist.

Finally, from Eqs. (2′), we deduce the pressure

$$p_\pm(x, y) = \int_{-\infty}^{+\infty} \sum_{k=1,2} \frac{\bar{J}_{k\pm}}{i\lambda A^2(1 - M^2 + s_k^2)} \times e^{\pm i\lambda s_k y}\, e^{-i\lambda x}\, d\lambda \tag{44}$$

In deducing this result we have also used Lebesque's theorem as well as the fact that $p(-\infty, y) = 0$, as it results from Eq. (1.11) in Ref. 1. On the basis of condition (50), we have

$$p(x, y) = -p(x, -y) \tag{44'}$$

We have seen that in the representation of the general solution (21–23), four functions are still to be determined. This will be made from the boundary conditions

$$v(x, \pm 0) = Y'(x) \qquad x\epsilon[-1, 1] \tag{45}$$

as well as from the conditions insuring the continuity of the magnetic field

$$\begin{aligned} h_x(x, +0) - h_x(x, -0) &= 0 \qquad \forall x \\ h_y(x, +0) - h_y(x, -0) &= 0 \qquad \forall x \end{aligned} \tag{46}$$

Taking account of (34) and (35), from (46) we deduce

$$\frac{s_1}{1 + s_1^2}(\bar{J}_{1+} + \bar{J}_{1-}) + \frac{s_2}{1 + s_2^2}(\bar{J}_{2+} + \bar{J}_{2-}) = 0 \tag{47}$$

$$\frac{1}{1 + s_1^2}(\bar{J}_{1+} - \bar{J}_{1-}) + \frac{1}{1 + s_2^2}(\bar{J}_{2+} - \bar{J}_{2-}) = 0 \tag{48}$$

On the other hand (45) implies

$$v(x, +0) - v(x, -0) = 0 \qquad \forall x \tag{45'}$$

whence with the aid of (30) and (31′) we deduce

$$\frac{s_1}{1 - M^2 + s_1^2}(\bar{J}_{1+} + \bar{J}_{1-}) + \frac{s_2}{1 - M^2 + s_2^2}(\bar{J}_{2+} + \bar{J}_{2-}) = 0 \tag{49}$$

System (47) and (49) is homogeneous and has the determinant different from zero. Hence,

$$\bar{J}_{k+} = -\bar{J}_{k-} \tag{50}$$

From (48) we get

$$(1 + s_1^2)\bar{J}_{2+} + (1 + s_2^2)\bar{J}_{1+} = 0 \tag{48'}$$

which shows that only one function $\bar{J}$ remains unknown. It will be determined from condition (45) in which we take account of (28) and (30).

We obtain the following integral equation:

$$\int_{-\infty}^{+\infty} F(\lambda)e^{-i\lambda x}\, d\lambda = Y'(x),\ x\epsilon[-1, 1] \tag{51}$$

where we have used the notations

$$F(\lambda) = f(i\lambda)\bar{J}_{1+}(\lambda)$$

$$f(i\lambda) = \frac{1}{i\lambda A^2} \times \left[\frac{s_1(1 + s_1^2)(1 - M^2 + s_2^2) - s_2(1 + s_2^2)(1 - M^2 + s_1^2)}{(1 + s_1^2)(1 - M^2 + s_1^2)(1 - M^2 + s_2^2)}\right] \tag{52}$$

Considering now that s_k are the roots of Eq. (15), we have

$$\begin{aligned} P_1 &= (1 + s_1^2)(1 + s_2^2) = A^2M^2[1 - (i\lambda/N)]^{-1} \\ P_2 &= (1 - M^2 + s_1^2)(1 - M^2 + s_2^2) = -M^2[1 - (i\lambda/N)]^{-1} \end{aligned} \tag{53}$$

These results make it possible to express the function $f(i\lambda)$ with the aid of the coefficients a, b, c. This function becomes thus known and different from zero. From (51), by imposing the condition that the integrand be real, there also results $\bar{J}_{1+}(\lambda) = \bar{J}_{1+}(-\lambda)$.

We have[3] the relation

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}\, t^n \tag{54}$$

where $H_n(x)$ are Hermite's polynomials. If in (54) we put $2t = -ix$, $x = \lambda$, we obtain

$$e^{-i\lambda x} = \sum_{n=0}^{\infty} \left(\frac{1}{2i}\right)^n \frac{x^n}{n!}\, H_n(\lambda)e^{-(x^2/4)} \tag{54'}$$

Since Hermite's polynomials form a complete system in the interval $(-\infty, +\infty)$, the solution of Eq. (51) may be represented as follows:

$$F(\lambda) = \sum_{m=0}^{\infty} A_m\, H_m(\lambda)e^{-\lambda^2} \tag{55}$$

where we have introduced the factor $\exp(-\lambda^2)$ in order to improve the series convergence. We introduce the expressions (54′) and (55) in the integral equation (51) and take into account the relations[3]:

$$\int_{-\infty}^{+\infty} H_m(\lambda)\, H_n(\lambda)e^{-\lambda^2}\, d\lambda = \begin{cases} 0, \text{ if } m \neq n \\ 2^n\pi^{1/2}n!, \text{ if } m = n \end{cases} \tag{56}$$

In this manner we get

$$\sum_{n=0}^{\infty} A_n x^n = i^n\pi^{-1/2}Y'(x)\exp\left(\frac{x^2}{4}\right),\ x\epsilon(-1, 1) \tag{57}$$

By assuming that the function $Y(x)$ is analytical we obtain

$$A_n = (i^n/n!\pi^{1/2})(d^n/dx^n)[Y'(x)\exp(x^2/4)]_{x=0} \tag{58}$$

The method just presented permits the extending of the results to the case in which the equation of the upper surface of the airfoil is not the same as that of the lower surface. Accordingly, we assume that

$$y_\pm = Y_\pm(x),\ x\epsilon[-1, 1] \tag{59}$$

In this case condition (43′) is replaced by

$$v(x, +0) - v(x, -0) = \begin{cases} [Y'(x)], \text{ if } x\epsilon[-1, 1] \\ 0, \text{ if } x\epsilon(-\infty, -1)U(1, \infty) \end{cases} \quad (60)$$

where we have denoted

$$[Y'(x)] = Y_+'(x) - Y_-'(x) \quad (61)$$

Taking into consideration (30) and (31′) and reversing the Fourier type integral which appears here, we deduce

$$\frac{s_1}{1 - M^2 + s_1^2}(\bar{J}_{1+} + \bar{J}_{1-}) + \frac{s_2}{1 - M^2 + s_2^2}(\bar{J}_{2+} + \bar{J}_{2-}) = \frac{1}{2\pi}\int_{-1}^{+1}[Y'(x)]e^{i\lambda x}\,dx = I(\lambda) \quad (62)$$

From (47) and (62) we obtain

$$\bar{J}_{1+} + \bar{J}_{1-} = -[1/s_1(1 + s_2^2)] \times [P_1P_2I(\lambda)/M^2(s_1^2 - s_2^2)]$$
$$\bar{J}_{2+} + \bar{J}_{2-} = [1/s_2(1 + s_1^2)][P_1P_2I(\lambda)/M^2(s_1^2 - s_2^2)] \quad (63)$$

P_1 and P_2 being the products defined in (53). Further, the solution of the problem may be simply obtained.

The previous method also permits the consideration of the problem of surface currents whose solution may be thus readily obtained.

References

[1] Dragos, L., "Theory of thin airfoils in magnetoaerodynamics," AIAA J. 2, 1223–1229 (1964).

[2] Schwartz, L., *Méthodes mathématiques pour les sciences physiques* (Hermann, Paris 1962), pp. 200–202.

[3] Courant, R. and Hilbert, D., *Methods of Mathematical Physics* (Springer-Verlag, Berlin, 1931), Vol. I, chap. 2,

A Comment on Hypersonic Viscous Interaction Theory

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New formulations of the hypersonic weak interaction theory (HWIT) and the hypersonic strong interaction theory (HSIT) for the case of uniform flow past a flat plate have been reported recently by the authors.[1,2] The characteristics of the boundary layers associated with such HWIT and HSIT flows are summarized and comparisons of these characteristics with those of "classical" compressible boundary layers are made. The definition of the boundary-layer outer-edge mass flow function, $\rho_e u_e$, in such hypersonic viscous interaction problems is examined.

The HWIT and HSIT flows considered are defined, in part, following Van Dyke,[3] by the criteria

$$M \gg 1 \qquad \tau \ll 1 \qquad K = M\tau \ll 1 \qquad \text{(HWIT)}$$
$$M \gg 1 \qquad \tau \ll 1 \qquad K = M\tau \gg 1 \qquad \text{(HSIT)} \quad (1a)$$

where M = freestream Mach number, τ = nondimensional thickness parameter of the viscous boundary layer. Further, as is pointed out in Ref. 3 and elsewhere, these criteria for the theories may be expressed more formally as:

$$M \to \infty \qquad \tau \to 0 \qquad K = M\tau \to 0 \qquad \text{(HWIT)}$$
$$M \to \infty \qquad \tau \to 0 \qquad K = M\tau \to \infty \qquad \text{(HSIT)} \quad (1b)$$

For L = characteristic length chosen so that $(x/L) = 0(1)$ in the region where the particular theory (HWIT or HSIT) is valid, and u_∞, T_∞, p_∞, ρ_∞ = the velocity in the x direction, temperature, pressure, and density, respectively, in the undisturbed region upstream of the flat plate, the appropriate HWIT and HSIT representations for the variables in the viscous boundary layers are:

$$\begin{aligned} (x/L) &= x_{BL} & (y/L) &= \tau y_{BL} \\ (u/u_\infty) &= u_{BL} + \ldots & (v/u_\infty) &= \tau v_{BL} + \ldots \\ (p/p_\infty) &= 1 + (M\tau)p_{BL} + \ldots \\ (T/T_\infty) &= (M^2)T_{BL} + \ldots \\ (\rho/\rho_\infty) &= (1/M^2)\rho_{BL} + \ldots \end{aligned} \quad \text{(2a) (HWIT)}$$

$$\begin{aligned} (x/L) &= x_{BL} & (y/L) &= \tau y_{BL} \\ (u/u_\infty) &= u_{BL} + \ldots & (v/u_\infty) &= \tau v_{BL} + \ldots \\ (p/p_\infty) &= (M^2\tau^2)p_{BL} + \ldots \\ (T/T_\infty) &= (M^2)T_{BL} + \ldots \\ (\rho/\rho_\infty) &= (\tau^2)\rho_{BL} + \ldots \end{aligned} \quad \text{(2b) (HSIT)}$$

With these representations, for a perfect gas having constant specific heats, a constant Prandtl number, σ, of order unity, and a viscosity coefficient obeying the law $(\mu/\mu_\infty) = (T/T_\infty)^\omega$, the leading terms in the equations of motion for the viscous boundary layers for hypersonic ($M \to \infty$), high Reynolds number ($R_L = \rho_\infty u_\infty L/\mu_\infty \to \infty$) flows past a flat plate are:

$$\frac{\partial}{\partial x_{BL}}(\rho_{BL}u_{BL}) + \frac{\partial}{\partial y_{BL}}(\rho_{BL}v_{BL}) = 0 \quad (3a)$$

$$\partial p_{BL}/dy_{BL} = 0 \qquad \text{i.e.} p_{BL} = {}_{BL}(x_{BL}) \quad (3b)$$

$$\rho_{BL}\left(u_{BL}\frac{\partial u_{BL}}{\partial x_{BL}} + v_{BL}\frac{\partial u_{BL}}{\partial y_{BL}}\right) + Z_{sw}\left(\frac{1}{\gamma}\right)\frac{\partial p_{BL}}{\partial x_{BL}} = \Lambda\frac{\partial}{\partial y_{BL}}\left(T_{BL}{}^\omega\frac{\partial u_{BL}}{\partial y_{BL}}\right) \quad (3c)$$

$$\rho_{BL}\left(u_{BL}\frac{\partial T_{BL}}{x\partial x_{BL}} + v_{BL}\frac{\partial T_{BL}}{x\partial y_{BL}}\right) - Z_{sw}\left(\frac{\gamma - 1}{\gamma x}\right)u_{BL} \times \frac{\partial p_{BL}}{\partial x_{BL}} = \Lambda\left\{\frac{1}{\sigma}\frac{\partial}{\partial y_{BL}}\left(T_{BL}{}^\omega\frac{\partial T_{BL}}{\partial y_{BL}}\right) + (\gamma - 1)T_{BL}{}^\omega\left(\frac{\partial u_{BL}}{\partial y_{BL}}\right)^2\right\} \quad (3d)$$

with

$$\rho_{BL}T_{BL} = 1 \qquad Z_{sw} = 0 \qquad \Lambda = \frac{M^{2(1+\omega)}}{R_L\tau^2} = 0(1) \qquad \text{(HWIT)}$$
$$\rho_{BL}T_{BL} = p_{BL} \qquad Z_{sw} = 1 \qquad \Lambda = \frac{M^{2\omega}}{R_L\tau^4} = 0(1) \qquad \text{(HSIT)} \quad (4)$$

The appropriate boundary conditions for the flow variables of these boundary layers (for both hypersonic interaction regimes) are (1) at the sharply defined[4] outer edge,‡ $y_{BL} \to$

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‡ Although Refs. 1 and 2 are not in total agreement with Stewartson's formulations of the HWIT and HSIT for flows past a flat plate,[4] there is complete agreement that, for both the HWIT and HSIT, the outer edges of the boundary layers are sharply defined.